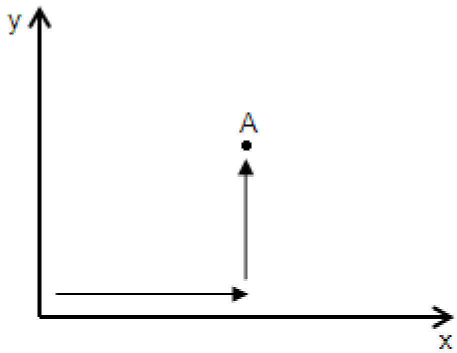
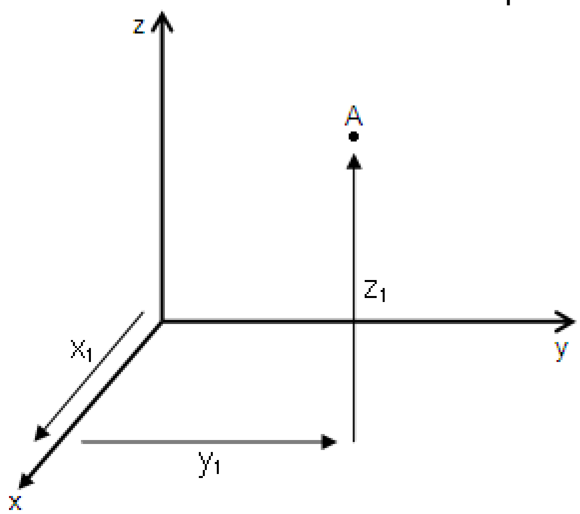
MATHEMATICS NOTES
S.4 THREE DIMENSIONAL GEOMETRY

A line has one dimension because to locate a point on it, we move in one direction.

A plane has two dimensions because to locate a point on a plane we use two coordinates (x, y).

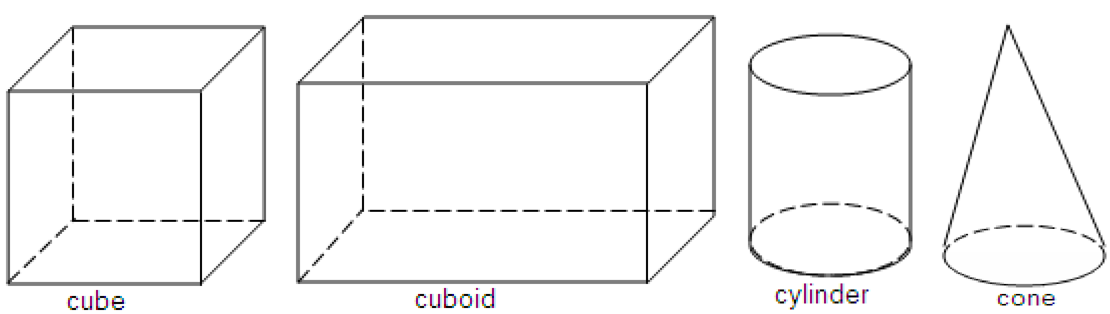


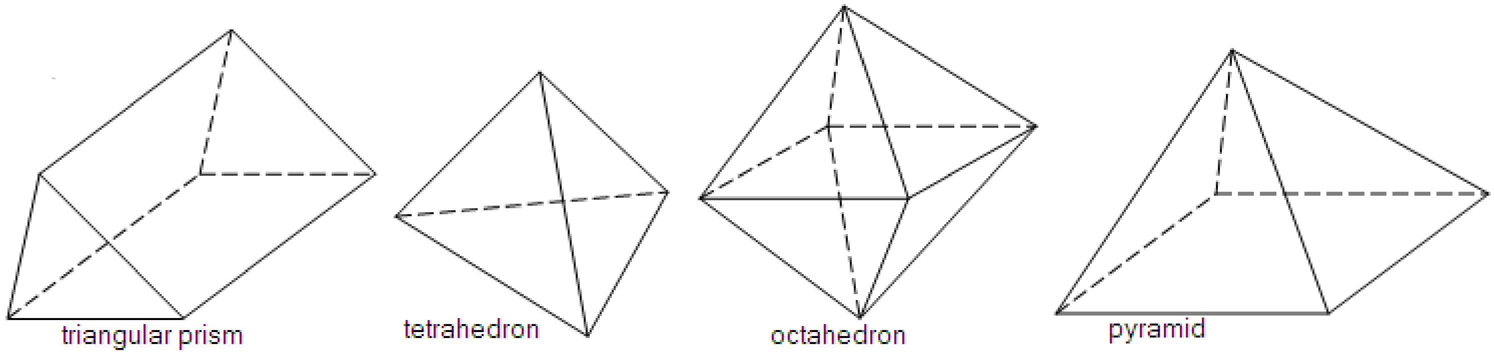
A solid has three dimensions because a point on it is located using three coordinates (x, y, z).



Geometrical properties of solids

Some of the common regular solids are cubes, cuboids, cylinders, pyramids, prisms, cones and spheres.

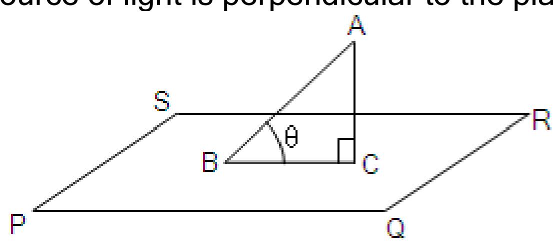




Solids may have faces, edges, and vertices. These faces, edges and vertices determine the geometrical properties of a solid. Solids with flat faces are called **Polyhedra** (singular, polyhedron). Polyhedron means many faces. A regular polyhedron consists of faces that are made of regular polygons.

Projection of a line on a plane

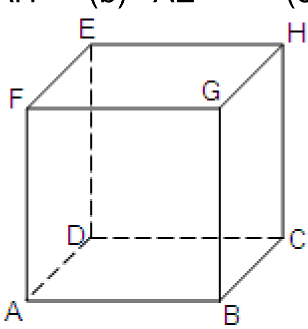
In the figure on the next page, AB is a line meeting plane PQRS at point B. The line is inclined to the plane at angle θ. Line AC is perpendicular to the plane from point A. Line BC is referred to as the **Projection** of line AB on plane PQRS. The projection of line AB on plane PQRS is the shadow of line AB when the source of light is perpendicular to the plane. In this case, the shadow of line AB is line BC.



Example 10.1

Below is a cube. State the projection of the following lines on plane ABCD.

(a) AH (b) AE (c) BE (d) BH

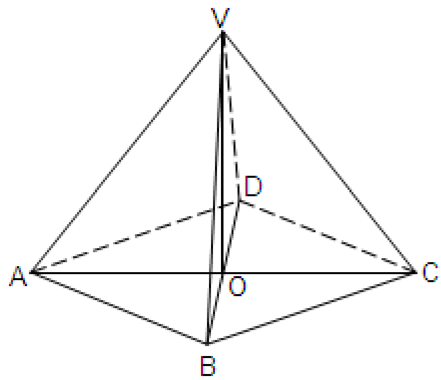


Solution

(a) AC (b) AD (c) BD (d) BC

Exercise 10.1

The figure below is a right pyramid with a square base, ABCD. Point O is the centre of the square.

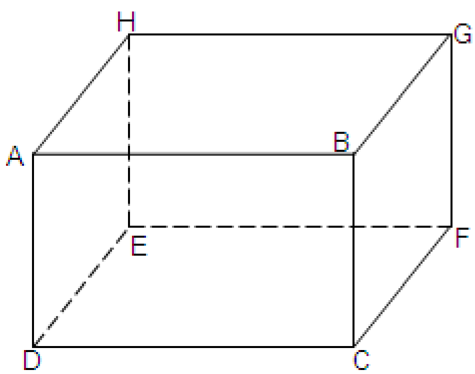


Use the figure to answer the following questions.

1. State the projection of the following in plane ABCD.
   (a) VA (b) VB (c) VC (d) VD
2. Name the lines that are perpendicular to VO.
3. Name the triangular planes that are perpendicular to each other.
4. State the projection of:
   (a) line VA on plane VDB
   (b) line VB on plane VAC.
   (c) line VC on plane VDB.

An angle between two lines

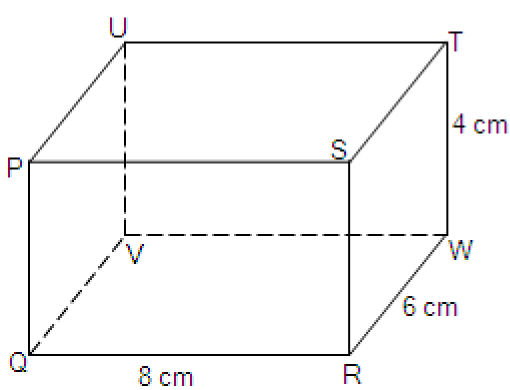
The figure below is a cuboid.



From the figure, we can identify and calculate the angles between two lines using Pythagoras' theorem and/or trigonometric ratios. For example, the angle between GC and CF is angle GCF.

Example 10.2

The diagram below shows a cuboid with dimensions 8 cm by 6 cm by 4 cm.

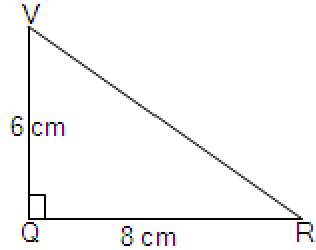


Find the angle between:

(a) VR and QR (b) SQ and QR.

Solution

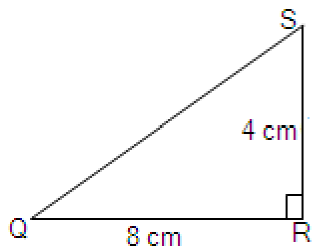
(a) VR and QR are on plane VQRW. We consider triangle VQR which is right-angled at Q. The required angle is VRQ.
We use the tangent ratio because the opposite and adjacent sides of angle VQR are given.



Thus, $\tan\angle VRQ = \frac{6}{8} = 0.7500$.

Therefore, $\angle$ VRQ = 36.9^0.

(b) SQ and QR are sides of the right-angled triangle SQR. The angle between lines SQ and QR is SQR.

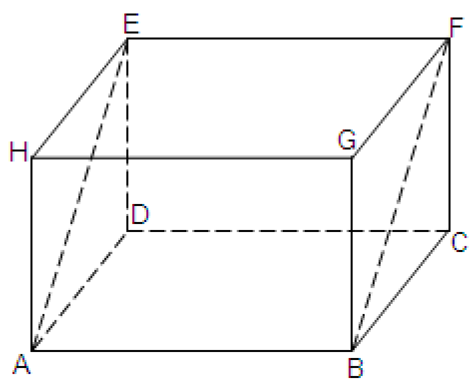


Thus, $\tan\ S\hat{Q}R = \frac{4}{8} = 0.5000$.

Therefore, $S\hat{Q}R = 26.6^0$

Skew lines

Lines that are not parallel and do not meet are called skew lines. The figure below is a cuboid.

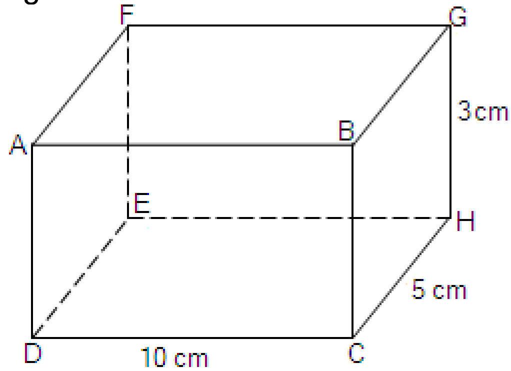


From the figure, lines AH and EF are skew. They are not parallel but do not meet.
Name other skew lines from the figure.

To find the angle formed by skew lines, one of the lines must be translated. For example, from the figure above, the angle between lines FB and EH is found by translating line FB onto EA on plane HADE. When this is done, point F coincides with point E and point B with A. Thus, the angle between the two lines is $H\hat{E}A$.

Example 10.3

The figure below is a cuboid. The dimensions of the cuboid are 10 cm by 5 cm by 3 cm.



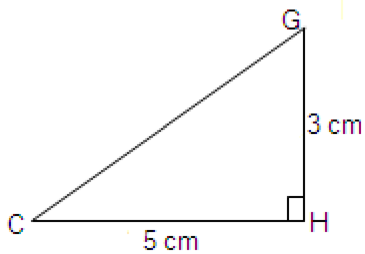
Find the angle between:

(a) Lines CG and DE.

(b) Lines FG and DB.

Solutions

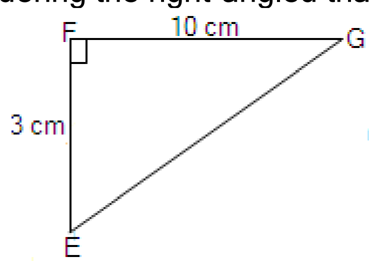
(a) CG and DE are skew lines. To find the angle between them we translate DE onto CH on plane BGHC. We consider triangle GCH which is right-angled at H. The required angle is $G\hat{C}H$



Thus, $\tan G\hat{C}H = \frac{3}{5}$ = 0.6000

Therefore, $G\hat{C}H$ = 31^0

(b) Lines FG and DB are skew lines. To find the angle between them, we translate DB onto plane EFGH. Point D coincides with point E and point B coincides with point G.
The angle between the lines FG and DB is formed at G, that is, $F\hat{G}E$.
Considering the right-angled triangle EFH, EF = 3 cm and FG = 10 cm.

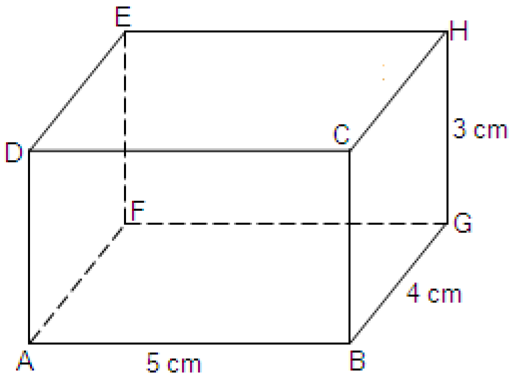


Thus, $\tan F\hat{G}E = \frac{3}{10}$ = 0.3000.

Therefore, $F\hat{G}E$ = 16.7^0.

Exercise 10.2

1. The figure below is a cuboid whose dimensions are 5 cm by 4 cm by 3 cm.



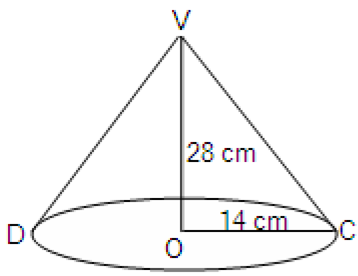
Find the angle between:

(a) Lines AH and AG.

(b) Lines AE and AG

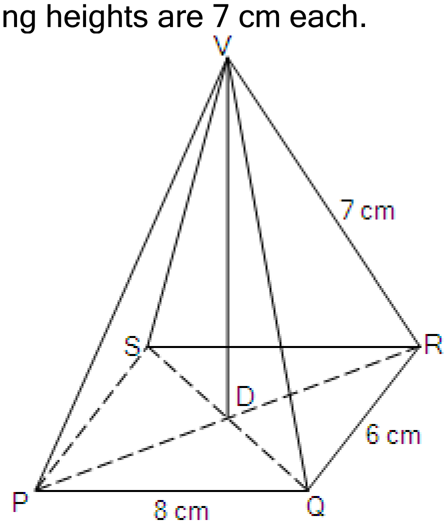
(c) Lines BH and AD.

2. The figure below is a cone whose diameter is 28 cm and its height is 28 cm.



Find the angle between the diameter and the line VC on the curved surface of the cone.

3. The figure below is a pyramid. The base of the pyramid is a rectangle whose dimensions are 8 cm by 6 cm. Its slanting heights are 7 cm each.



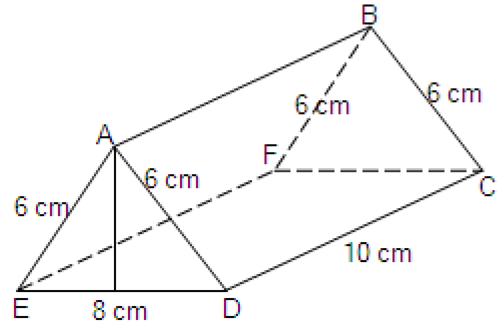
Find the angles between:

(a) Lines VQ and QS.

(b) Lines SQ and PQ.

(c) Lines QR and VP.

4. The figure below is a triangular prism. Its dimensions are indicated on the figure.



Find the angle between:

(a) Lines AE and AD.

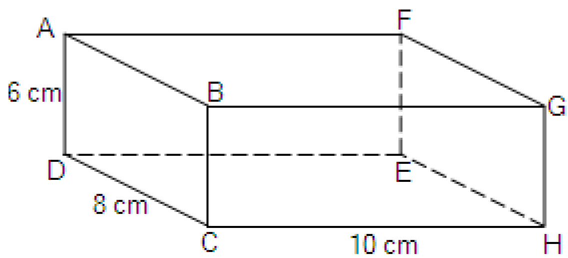
(b) Lines EB and EC.

The length of a line on a solid

To find the length of a line in three dimensions, we identify the plane containing the line. The plane may be a triangle, rectangle, parallelogram or any other plane shape. We then apply Pythagoras' theorem or trigonometric ratios to find the lengths of the lines.

Example 10.4

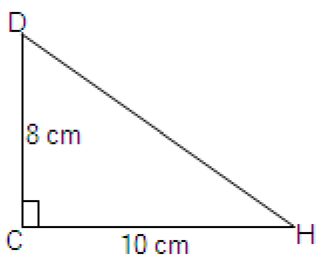
The figure given is a cuboid whose dimensions are 6 cm by 8 cm by 10 cm.



Find the length of line DG.

Solution

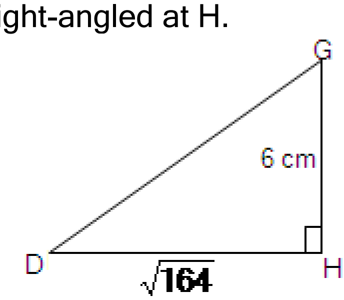
Consider ΔHDC and ΔGDH which are right angled triangles. ΔHDC is right angled at C.



$DH^2 = 8^2 + 10^2 = 164$

$DH = \sqrt{164} = 12.81$ cm (to 2 d.p.)

Triangle GDH is right-angled at H.



$DG^2 = DH^2 + HG^2$

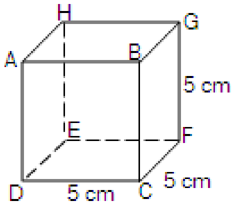
$= 164 + 36 = 200$

$DG = \sqrt{200} = 10\sqrt{2}$ cm.

$= 14.14$ cm (to 2 d.p.)

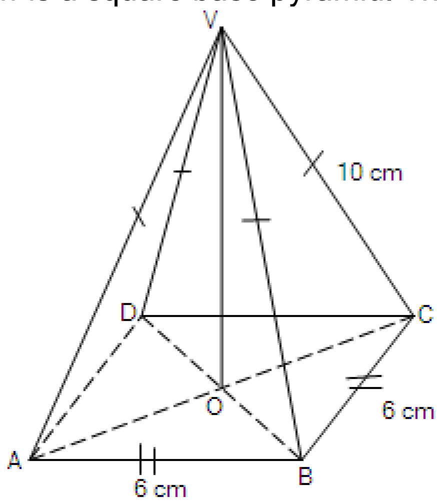
Exercise 10.3

1. The figure below is a cube of side 5 cm.



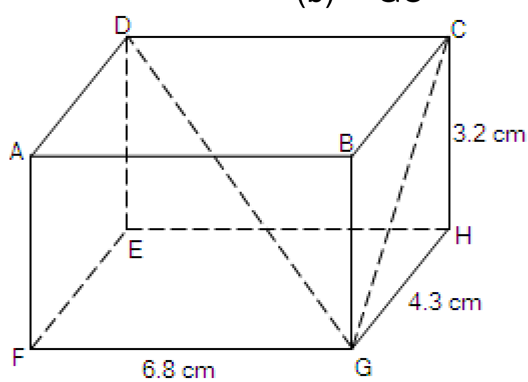
Find the length of line: (a) CE, (b) CH, (c) DH.

2. The figure below is a square base pyramid. The faces of the pyramid are isosceles triangles.

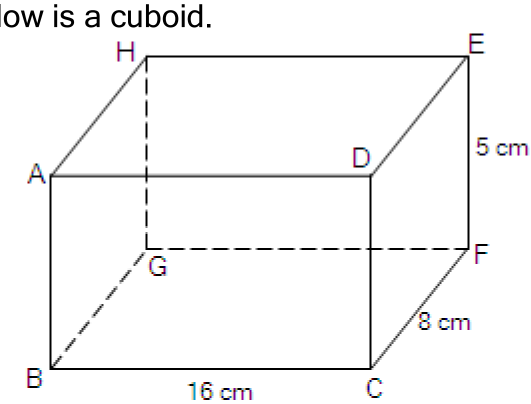


   (a) Find the length of the diagonals of the base.
   (b) Find the height of the pyramid.

3. In the cuboid shown below, find the length of the line:
   (a) DG   (b) GC



4. The figure below is a cuboid.



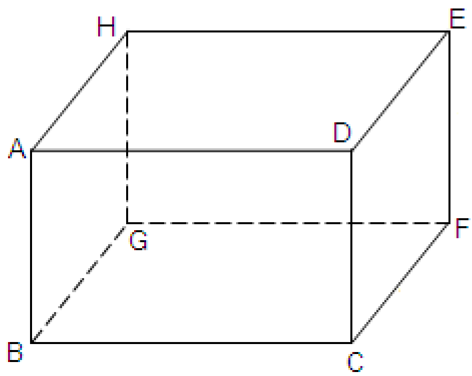
   Find the length of line:
   (a) BF   (b) BE.

Angle between a line and a plane

The angle between a line and a plane is the angle between the line and its projection on the plane.

Example 10.5

Below is a cuboid.



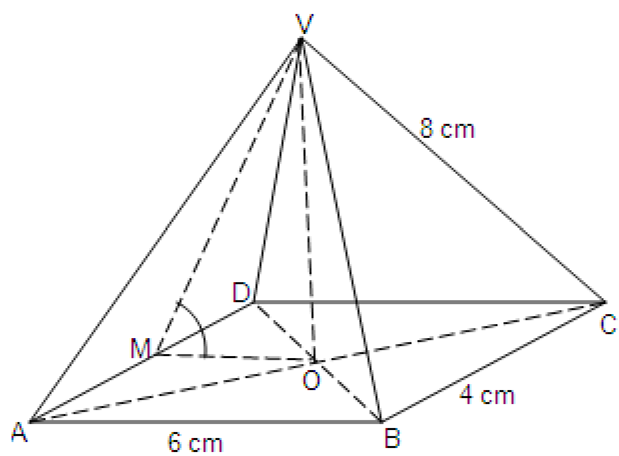
State the projection of line BE on plane BCFG and the angle between line BE and plane BCFG.

Solution

The projection of line BE on plane BCFG is BF. Thus, the angle between line BE and plane BCFG is $E\hat{B}F$.

Example 10.6

The figure below is a pyramid VABCD. V is the vertex of the pyramid and ABCD is the rectangular base of the pyramid. AB = 6 cm, BC = 4 cm and the slant height, VC = 8 cm. O is the intersection of the diagonals of the base.



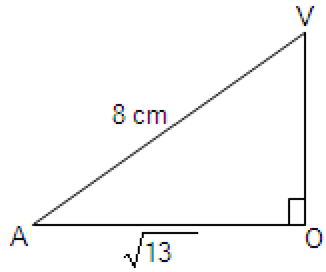
(a) Find the angle between VA and the base.
(b) If M is the midpoint of AD, find the angle between VM and plane ABCD

Solution

(a) AO is the projection of VA on the plane ABCD. In triangle VAO, the angle between line VA and plane ABCD is $V\hat{A}O$.

$$AO = \frac{1}{2}AC = \frac{1}{2}\sqrt{(36+16)}$$

$$= \frac{1}{2}\sqrt{52} = \sqrt{13} = 3.61 \text{ cm.}$$

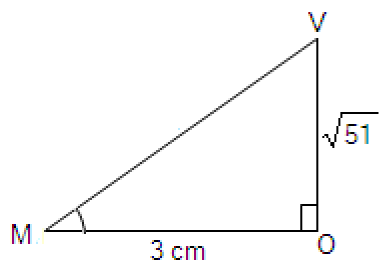


$\text{Cos } V\hat{A}O = \frac{\sqrt{13}}{8} \approx 0.4507$

From mathematical tables or calculator, we find an angle whose cosine is 0.4507.

$V\hat{A}O = 63.21^0$ (to 2 d.p.)

(b) The angle between line VN and the plane ABCD is $V\hat{M}O$. The projection of line VM on plane ABCD is MO.



$VM^2 = MO^2 + VO^2$

From triangle VAO above, VO = $\sqrt{VA^2 - AO^2}$

$= \sqrt{(64 - 13)}$

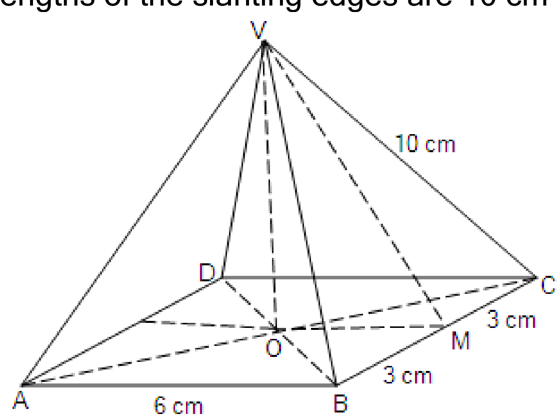
$= \sqrt{51}$

$\text{Tan } V\hat{M}O = \frac{\sqrt{51}}{3} = 2.380$

$V\hat{M}O = 67.21^0$ (to 2 d.p.)

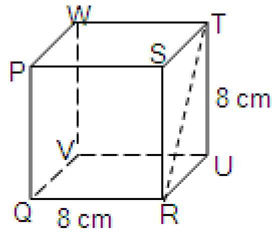
Exercise 10.4

1. The figure below is a square based pyramid. The triangular faces of the pyramid are isosceles triangles. The lengths of the slanting edges are 10 cm each.



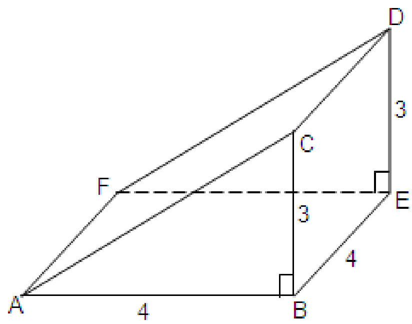
(a) Find the angle between line VA and plane ABCD.
(b) If M is the midpoint of BC, find the angle between VM and plane ABCD.

2. The figure below is a cube of side 8 cm.



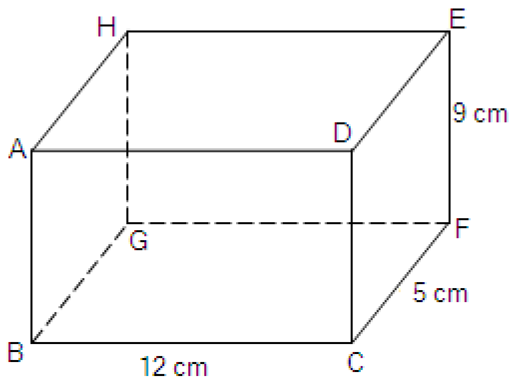
(a) Name the projection of line QS on plane QRUV.
(b) Name any two pairs of skew lines.
(c) Find the angle between lines RT and WV.

3. The figure below is a triangular prism.



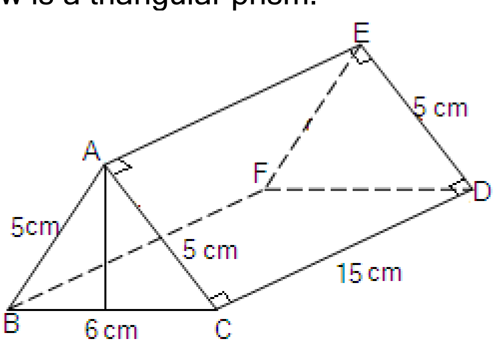
(a) Calculate the length of line: (i) AC (ii) AE (iii) AD
(b) Calculate the angle between lines FC and FB.

4. The figure below is a cuboid.



(a) Calculate the length of line:
 (i) BF (ii) FH (iii) BH
 (iv) CH
(b) Calculate the angles between lines:
 (i) BE and BF
 (ii) BD and BG
 (iii) GE and EC.

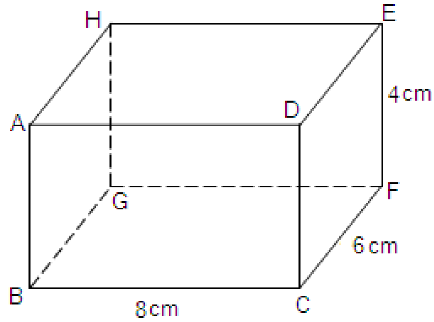
5. The figure below is a triangular prism.



Calculate:
(a) the length of line CE

(b) the angle between lines CE and EA.

6. The figure below is a cuboid.



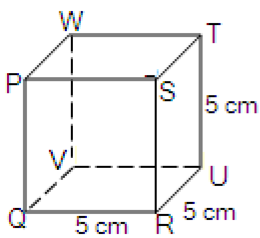
(a) Name the projection of line BE on plane ABGH

(b) Find the length of line:

(i) BF (ii) BE

(c) Find the angle between lines BE and BF.

7. The figure below is a cube.



(a) Name:

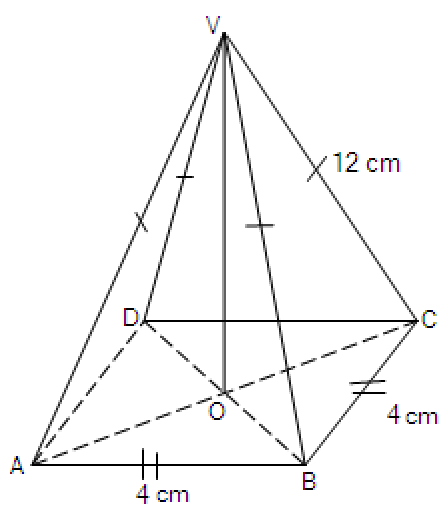
(i) the projection of line QS on plane QRUV.

(ii) The projection of line RV on plane RSTU.

(b) Find the angle between lines QT and WQ.

(c) Calculate the angle between line QS and plane QRUV.

8. The figure below is a square based pyramid with vertex V. Point O is the intersection of diagonals AC and BD.



(a) Name the projection of line VB on plane ABCD.

(b) Find the angle between line VA and line AC

(c) Calculate the angle between line VB and plane ABCD.

9. The figure below is a triangular prism. Its base is a right-angled triangle.

(a) Calculate the length of line:

(i) QS (ii) UQ (iii) UR

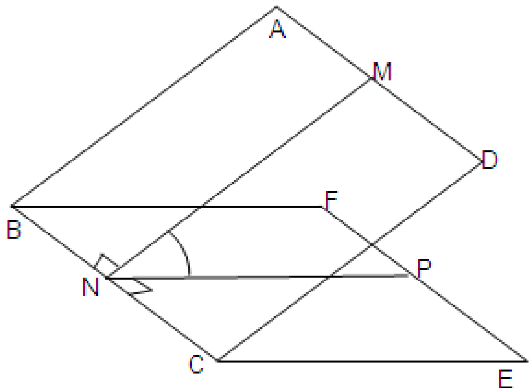
(b) Calculate:

(i) $U\hat{R}S$ (ii) $U\hat{Q}S$.

Angle between two planes

The angle between two intersecting planes is the angle between two lines, one in each plane, such that the two lines meet at a point on the line common to the planes, and both lines are at right angles to the common line.

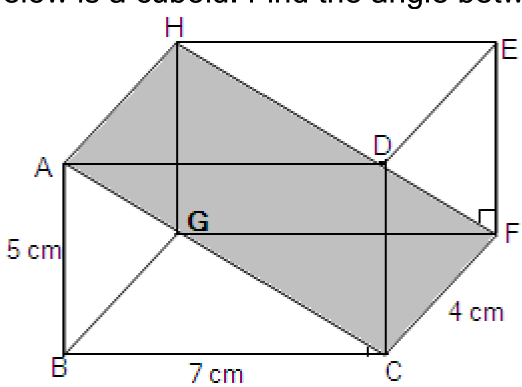
The diagram below shows two intersecting planes ABCD and BCEF.



The line of intersection of the two planes is BC. Lines MN and NP are perpendicular to line BC at N. The angle between the two planes is $M\hat{N}P$.

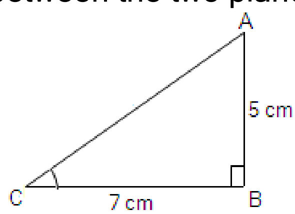
Example 10.7

The figure below is a cuboid. Find the angle between planes BCFG and ACFH.



Solution

The angle between the two planes is $A\hat{C}B$.



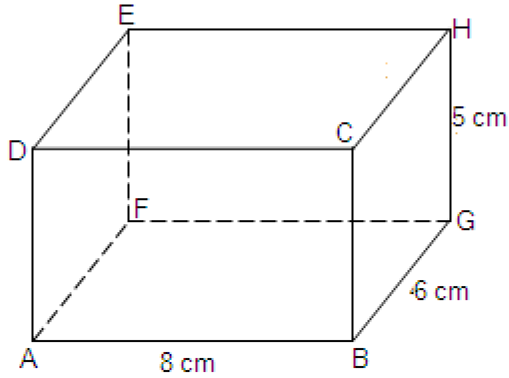
From triangle ABC,

$\text{Tan } A\hat{C}B = \frac{5}{7} \approx 0.7143$

$A\hat{C}B = 35.54^0$

Exercise 10.5

1. The figure below is a cuboid.



From the figure find the angles between planes:

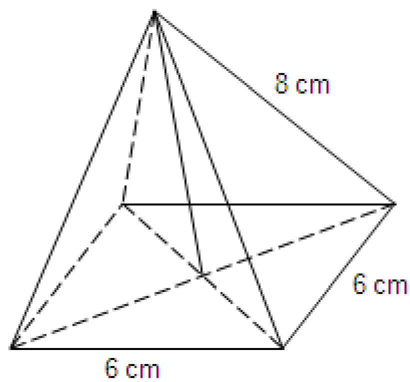
(a) ABGF and AFHC.

(b) BGED and DEHC.

(c) ABCD and BHEA.

(d) DEGB and BGHC.

2. The figure below shows a right pyramid on a square base. The slanting edges are 8 cm long each.



Find:

(a) the height of the pyramid.

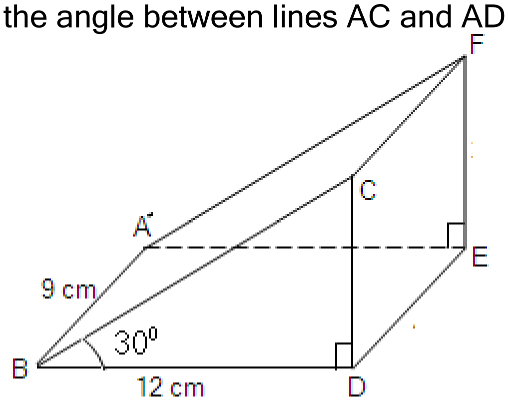
(b) the angle the slanting edges make with the base.

3. The following figure is a triangular prism.

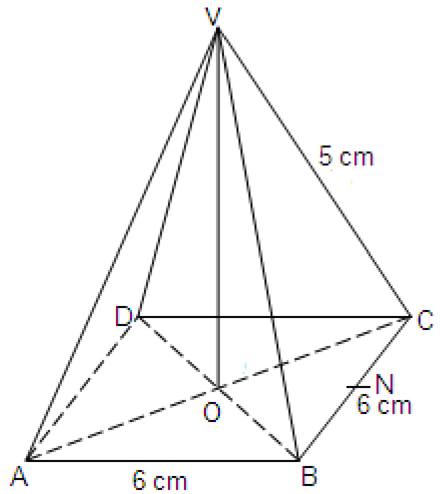
Find:

(a) height CD.

(b) the angle between lines AC and AD.



4. The following figure shows a square based pyramid of side 6 cm. Point N is the midpoint of BC. The slanting edges are 5 cm each.



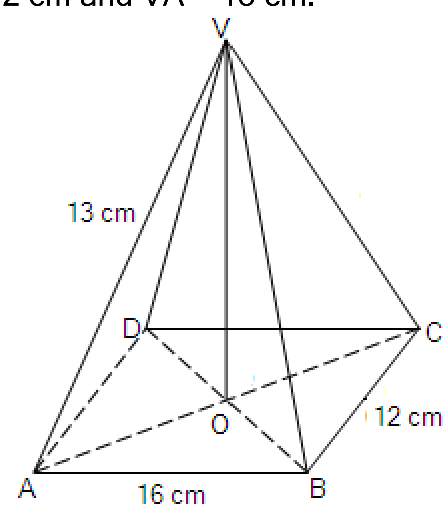
Find:
(a) the height of the pyramid.
(b) The angle between planes VBC and ABCD.

5. The faces of a square based pyramid are isosceles triangles. The sides of the square base are 12 cm. The lengths of the slanting edges are 10 cm each.
(a) Calculate the height of the pyramid.
(b) Find the angle between the slanting edges and the base.

6. The height of a circular cone is 35 cm and the radius of the base is 21 cm. Find:
(a) the length of the slanting height.
(b) the angle between the slanting height and the base.

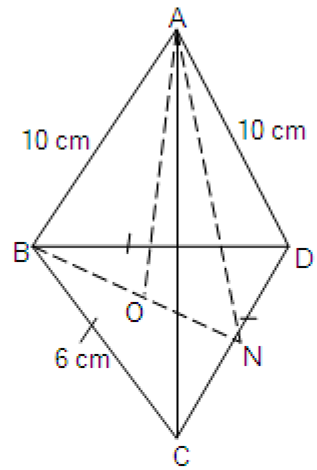
7. A pyramid has a rectangular base, 12 cm long and 9 cm wide. Its slanting faces are isosceles triangles whose edges are 15 cm long. Find:
(a) the height of the pyramid.
(b) the angle between a slanting edge and the base.
(c) the angle between the triangular faces and the base.

8. The figure below is a pyramid on a rectangular base. AB = 16 cm, BC = 12 cm and VA = 13 cm.



Find:
(a) the length of the line BD.
(b) the height of the pyramid
(c) the angle between VB and the base.
(d) the angle between plane VBC and the base.

9. The figure below is a tetrahedron. The base is an equilateral triangle of side 6 cm. Point N is the midpoint of CD. The slanting edges of the tetrahedron are 10 cm each. AO is the height of the tetrahedron.



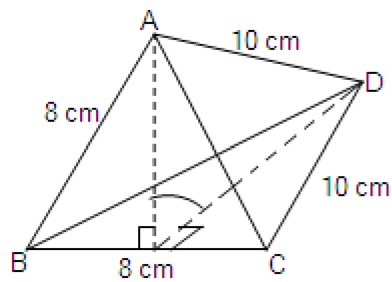
Find:

(a) the height of the tetrahedron.

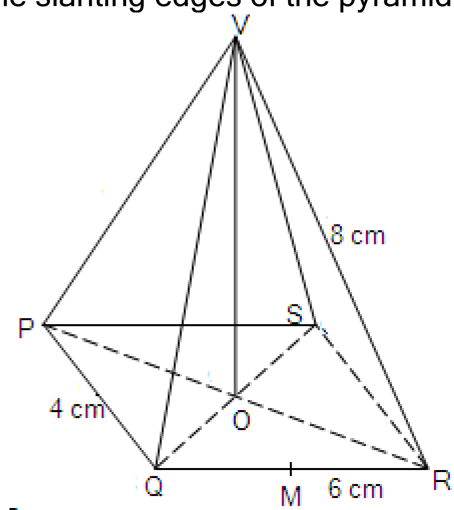
(b) the angle between line AD and plane BCD.

(c) the angle between planes ACD and BCD.

10. The following figure is a tetrahedron in which AB = BC = AC = 8 cm and AD = DC = BD = 10 cm. Find the angle between planes ABC and DBC.



11. The following figure is a right pyramid with a rectangular base, PQRS, and vertex V. PQ = 4 cm and QR = 6 cm. The slanting edges of the pyramid are 8 cm each. Point M is the midpoint of QR.



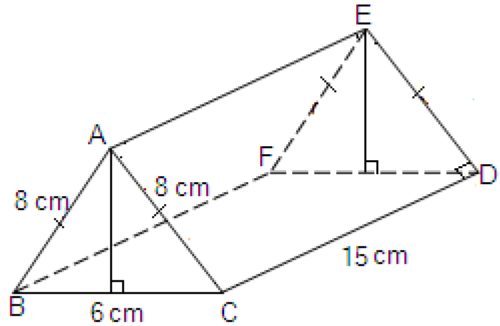
Calculate:

(a) the length of line VO.

(b) the angle between lines VS and SQ.

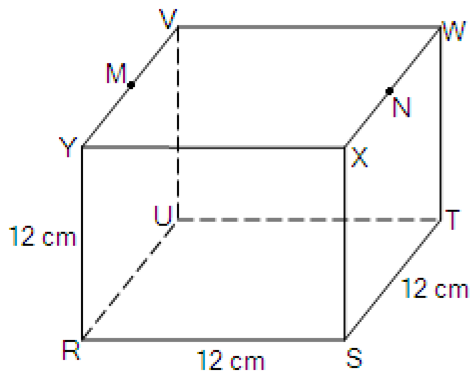
(c) the angle between line VM and plane PQRS.

12. The figure below is a triangular prism. The cross section of the prism is an isosceles triangle.



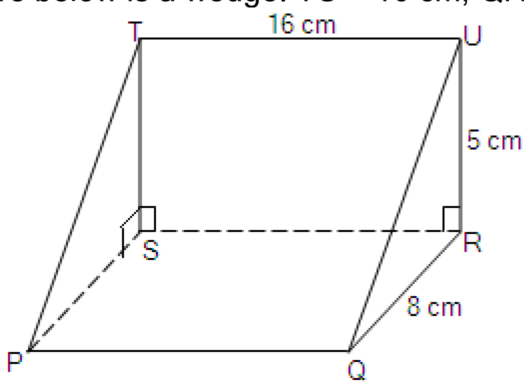
Calculate:
(a) the angle between lines AC and BC.
(b) The angle between lines CF and CD.
(c) The angle between planes ABFE and ACDE.

13. The figure below is a cube. Points M and N are the midpoints of YV and XW respectively.



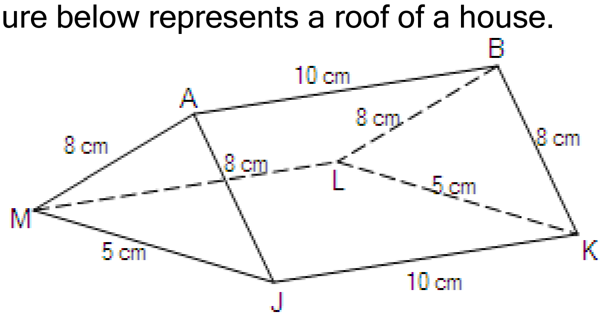
Find:
(a) the length of line SN.
(b) the angle between planes MNSR and UTSR.
(c) the angle between planes VWSR and MNSR.
(d) the angle between VU and NS.

14. The figure below is a wedge. TU = 16 cm, QR = 8 cm and RU = 5 cm.



(a) Write down the projection of line TQ onto plane PQRS.
(b) Find the angle between line TQ and plane PQRS.
(c) Find the angle between planes PQUT and STUR.

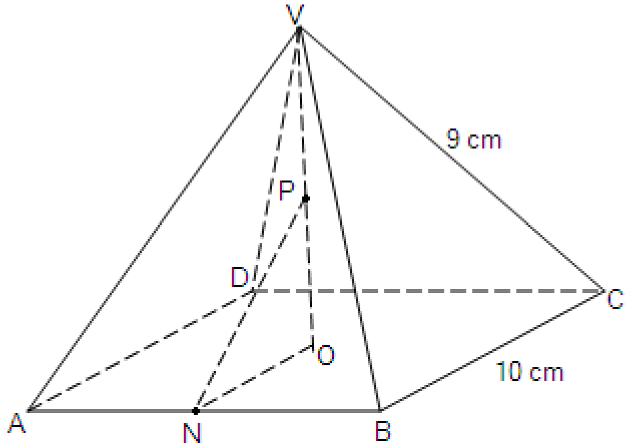
15. The figure below represents a roof of a house.



Find:

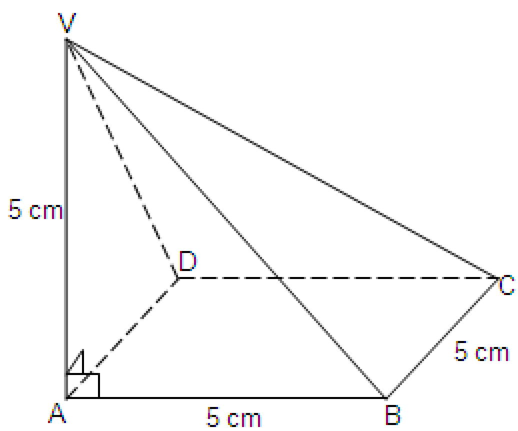
(a) the height of the roof,
(b) the angle between planes AMJ and MJKL.

16. The figure below is a right pyramid, VABCD. The base is a square, 10 cm each side. The vertex is vertically above the centre of the square.
VA = VB = VC = VD = 9 cm. Points P and N are the midpoints of VO and AB respectively.



Find:
(a) the length of the line VO.
(b) the angle between lines VA and AC.
(c) the projection of line PN on plane ABCD
(d) the angle between line PN and plane ABCD.
(e) the angle between planes VBC and ABCD.

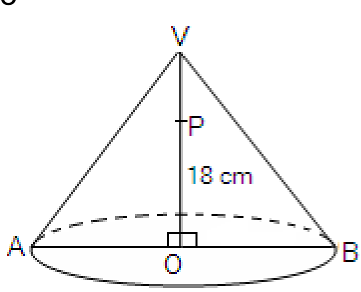
17. The figure below is a pyramid with a square base, ABCD, of length 5 cm. Edge VA is 5 cm long and perpendicular to the base at A.



(a) Name the projection of line VC on plane ABCD.
(b) Find the length of edge VC.
(c) Find the angle between edge VC and plane ABCD.

18. The following figure is a cone whose base radius is 14 cm. The perpendicular height, VO, is 18 cm. Point P is such that

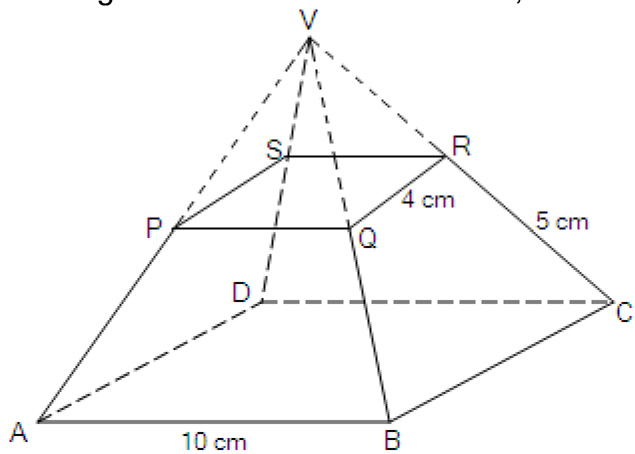
$OP = \frac{2}{3}OV$.



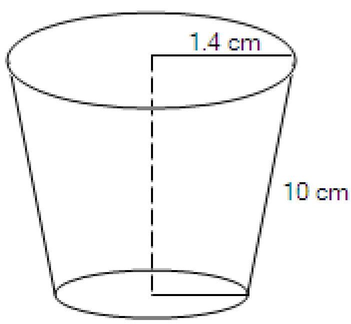
(a) Find the lengths of lines VB and PB.

(b) Find the angle between lines:
   (i) VB and BA.
   (ii) AP and AO.

19. The following diagram shows a frustum of a square based pyramid. The base, ABCD, is a square of side 10 cm long. The top, PQRS, is a square, 4 cm a side, and each of the short edges of the frustum is 5 cm.
Determine:
(a) the height of the frustum.
(b) The angle between AR and the base, ABCD.



20.



The figure represents a solid frustum cut off from a cone of height 4.2 cm. The radii of the circular ends are 1.4 cm and 0.7 cm. The slanting height of the frustum is 10 cm.
Calculate:
(a) the volume of the frustum.
(b) the surface area of the frustum.